Some Results On Intuitionistic Fuzzy H_v-subgroups

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Abstract-In this paper, we apply the concept of intuitionistic fuzzy set to H_v-groups. The notion of an intuitionistic fuzzy H_v-subgroup of an H_v-group is introduced and some related properties are investigated. Characterizations of intuitionistic fuzzy H_v-subgroups are given.

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1. Introduction

The concept of hyperstructure was introduced in 1934 by Marty [10]. Hyperstructures have many applications to several branches of pure and applied sciences. Vougiouklis [11] introduced the notion of H_v-structures, and Davvaz [5] surveyed the theory of H_v-structures. After the introduction of fuzzy sets by Zadeh [13], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [2, 3].

In [4] Biswas applied the concept of intuitoinistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. In [8] Kim et al. introduced the notion of fuzzy subquasigroups of a quasigroup. In [9] Kim and Jun introduced the concept of fuzzy ideals of a semigroup. B. Davvaz et al. [6] introduced the notion of an intuitionistic fuzzy H_v-submodule of an H_v-module. This paper continues this line of research for fuzzy H_v-subgroup of H_v-group.

The paper is organized as follows: in section 2 some fundamental definitions on H_v-structures and fuzzy sets are explored, in section 3 we define intuitionistic fuzzy H_v-subgroups and establish some useful theorems.

2. Basic Definitions

We first give some basic definitions for proving the further results.

Definition 2.1. [6] Let X be a non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy set in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by

$$\mu^c(x) = 1 - \mu(x) \quad \forall x \in X.$$

Definition 2.2. [6] Let f be a mapping from a set X to a set Y. Let μ be a fuzzy set in X and λ be a fuzzy set in Y. Then the inverse image $f^{-1}(\lambda)$ of λ is a fuzzy set in X defined by

$$f^{-1}(\lambda)(x) = \lambda(f(x)) \quad \forall x \in X.$$

The image $f(\mu)$ of μ is the fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

For all $y \in Y$.

Definition 2.3. [6] An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership and degree of non membership of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$. We shall use the symbol $A = \{\mu_A, \lambda_A\}$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

Definition 2.4. [6] Let $A = \{\mu_A, \lambda_A\}$ and $B = \{\mu_B, \lambda_B\}$ be intuitionistic fuzzy sets in X. Then

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x)$ *and* $\lambda_A(x) \le \lambda_B(x)\ \forall x \in X$,
(2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) : x \in X\}$,
(3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) : x \in X\}$,
(4) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) : x \in X\}$,
(5) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) : x \in X\}$,
(6) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) : x \in X\}$

Definition 2.5. [12] Let G be a non–empty set and $* : G \times G \to \wp^*(G)$ be a hyperoperation, where $\wp^*(G)$ is the set of all the non-empty subsets of G. Where $A * B = \bigcup_{a \in A, b \in B} a * b,\ \forall A, B \subseteq G$.

The $*$ is called weak commutative if $x * y \cap y * x \ne \phi,\quad \forall x, y \in G$.

The $*$ is called weak associative if $(x * y) * z \cap x * (y * z) \ne \phi,\quad \forall x, y, z \in G$.

A hyperstructure $(G, *)$ is called an H_v–group if

(i) $*$ is weak associative.

(ii) $a * G = G * a = G,\quad \forall a \in G$ (Reproduction axiom).

Definition 2.6. [7] Let G be a hypergroup (or H_v-group) and let μ be a fuzzy subset of G. Then μ is said to be a fuzzy subhypergroup (or fuzzy H_v-subgroup) of G if the following axioms hold:

$(i)\ \min\{\mu(x), \mu(y)\} \le \inf_{\alpha \in x * y} \{\mu(\alpha)\},\quad \forall x, y \in G$

(ii) For all $x, a \in G$ there exists $y \in G$ such that $x \in a * y$ and $\min\{\mu(a), \mu(x)\} \le \{\mu(y)\}$.

3. Intuitionistic fuzzy H_v-subgroup

In this section we give the definition of intuitionistic fuzzy H_v-subgroup and prove some related results.

Definition 3.1. Let G be a hypergroup (or H_v-group). An intuitionistic fuzzy set $A = \{\mu_A, \lambda_A\}$ of G is called intuitionistic fuzzy subhypergroup (or intuitionistic fuzzy H_v-subgroup) of G if the following axioms hold:

$(i)\ \min\{\mu_A(x), \mu_A(y)\} \le \inf\{\mu_A(z) : z \in x \cdot y\},\quad \forall x, y \in G$

(ii) For all $x, a \in G$ there exists $y \in G$ such that $x \in a \cdot y$ and $\min\{\mu_A(a), \mu_A(x)\} \le \{\mu_A(y)\}$

$(iii)\ \sup\{\lambda_A(z) : z \in x \cdot y\} \le \max\{\lambda_A(x), \lambda_A(y)\},\quad \forall x, y \in G$

(iv) For all $x, a \in G$ there exists $y \in G$ such that $x \in a \cdot y$ and $\{\lambda_A(y)\} \le \max\{\lambda_A(a), \lambda_A(x)\}$

Lemma 3.2. If $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G, then so is $\Box A = \{\mu_A, \mu^c{}_A\}$.

Proof. It is sufficient to show that $\mu^c{}_A$ satisfies the conditions (iii) and (iv) of Definition 3.1. For $x, y \in G$ we have

$$\min\{\mu_A(x), \mu_A(y)\} \le \inf\{\mu_A(z) : z \in x \cdot y\}$$

and so

$$\min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \le \inf\{1 - \mu_A^c(z) : z \in x \cdot y\}$$

Hence

$$\min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \le 1 - \sup\{\mu_A^c(z) : z \in x \cdot y\}$$

Which implies

$$\sup\{\mu_A^c(z) : z \in x \cdot y\} \le 1 - \min\{1 - \mu_A^c(x), 1 - \mu_A^c(y)\}$$

Therefore

$$\sup\{\mu_A^c(z) : z \in x \cdot y\} \le \max\{\mu_A^c(x), \mu_A^c(y)\}$$

And thus the condition (iii) of Definition 3.1 is valid.

Now, let $a, x \in G$. Then there exist $y \in G$ such that $x \in a \cdot y$ and

$$\min\{\mu_A(a), \mu_A(x)\} \le \{\mu_A(y)\}$$

It follows that

$$\min\{1 - \mu_A^c(a), 1 - \mu_A^c(x)\} \le \{1 - \mu_A^c(y)\}$$

$$\mu_A^c(y) \le 1 - \min\{1 - \mu_A^c(a), 1 - \mu_A^c(x)\}$$

So that

$$\{\mu_A^c(y)\} \le \max\{\mu_A^c(a), \mu_A^c(x)\}$$

Hence the condition (iv) of Definition 3.1 is satisfied.

Lemma 3.3. If $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G, then so is $\Diamond A = \{\lambda_A^c, \lambda_A\}$.

Proof. The proof is similar to the proof of Theorem 3.2.

Combining the above two lemmas it is not difficult to verify that the following theorem is valid.

Theorem 3.4. $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy H_v-subgroup of G.

Corollary 3.5. $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G if and only if μ_A and λ_A^c are intuitionistic fuzzy H_v-subgroup of G.

Definition 3.6. For any $t\in[0,1]$ and a fuzzy set μ in G, the set $U(\mu;t)=\{x\in G:\mu(x)\geq t\}$ and $L(\mu;t)=\{x\in G:\mu(x)\leq t\}$ is called an upper and lower t-level cut of μ.

Theorem 3.7. If $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G, then the sets $U(\mu_A;t)$ and $L(\lambda_A;t)$ are H_v-subgroup of G for every $t\in \operatorname{Im}(\mu_A)\cap\operatorname{Im}(\lambda_A)$.

Proof. Let $t\in \operatorname{Im}(\mu_A)\cap\operatorname{Im}(\lambda_A)\subseteq[0,1]$ and let $x,y\in U(\mu_A;t)$.

Then $\mu_A(x)\geq t$ and $\mu_A(y)\geq t$ and so
$\min\{\mu_A(x),\mu_A(y)\}\geq t$.
It follows from the condition (i) of Definition 3.1 that
$\inf\{\mu_A(z):z\in x\cdot y\}\geq t$.
Therefore $z\in U(\mu_A;t)$ for all $z\in x\cdot y$ and so
$x\cdot y\in U(\mu_A;t)$.
Hence $a\cdot U(\mu_A;t)\subseteq U(\mu_A;t)$ and
$U(\mu_A;t)\cdot a\subseteq U(\mu_A;t)$ for all $a\in U(\mu_A;t)$.
Now, let $x\in U(\mu_A;t)$.
Then there exist $y\in G$ such that $x\in a\cdot y$ and
$\min\{\mu_A(x),\mu_A(a)\}\leq\min\{\mu_A(y)\}$.
Since $x,a\in U(\mu_A;t)$, we have
$t\leq\min\{\mu_A(x),\mu_A(a)\}$ and so
$t\leq\min\{\mu_A(y)\}$, which implies $y\in U(\mu_A;t)$.
This proves that $U(\mu_A;t)\subseteq a\cdot U(\mu_A;t)$ and
$U(\mu_A;t)\subseteq U(\mu_A;t)\cdot a$.
If $x,y\in L(\lambda_A;t)$, then
$\max\{\lambda_A(x),\lambda_A(y)\}\leq t$.
It follows from the condition (iii) of Definition 3.1 that $\sup\{\lambda_A(z):z\in x\cdot y\}\leq t$.

Therefore for all $z\in x\cdot y$ we have $z\in L(\lambda_A;t)$, so $x\cdot y\subseteq L(\lambda_A;t)$. Hence for all $a\in L(\lambda_A;t)$ we have $a\cdot L(\lambda_A;t)\subseteq L(\lambda_A;t)$ and
$L(\lambda_A;t)\cdot a\subseteq L(\lambda_A;t)$. Now, let $x\in L(\lambda_A;t)$.
Then there exist $y\in G$ such that $x\in a\cdot y$
and $\max\{\lambda_A(y)\}\leq\max\{\lambda_A(a),\lambda_A(x)\}$.
Since $x,a\in L(\lambda_A;t)$, we have

$$\max\{\lambda_A(a),\lambda_A(x)\}\leq t$$

and so

$$\max\{\lambda_A(y)\}\leq t.$$

Thus

$$y\in L(\lambda_A;t).$$

Hence

$$L(\lambda_A;t)\subseteq a\cdot L(\lambda_A;t)$$

And

$$L(\lambda_A;t)\subseteq L(\lambda_A;t)\cdot a.$$

Theorem 3.8. If $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy set in G such that all non empty level sets $U(\mu_A;t)$ and $L(\lambda_A;t)$ are H_v-subgroup of G, then $A=\{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G.

Proof. Assume that all non empty level sets $U(\mu_A;t)$ and $L(\lambda_A;t)$ are H_v-subgroup of G. If $t_0=\min\{\mu_A(x),\mu_A(y)\}$ and $t_1=\max\{\lambda_A(x),\lambda_A(y)\}$ for $x,y\in G$, then $x,y\in U(\mu_A;t_0)$ and $x,y\in L(\lambda_A;t_1)$. So $x\cdot y\subseteq U(\mu_A;t_0)$ and $x\cdot y\subseteq L(\lambda_A;t_1)$.
Therefore for all $z\in x\cdot y$ we have $\mu_A(z)\geq t_0$ and $\lambda_A(z)\leq t_1$, that is,
$\inf\{\mu_A(z):z\in x\cdot y\}\geq\min\{\mu_A(x),\mu_A(y)\}$
And
$\sup\{\lambda_A(z):z\in x\cdot y\}\leq\max\{\lambda_A(x),\lambda_A(y)\}$
Which verify the conditions (i) and (iii) of Definition 3.1.

Now, If $t_2=\min\{\mu_A(a),\mu_A(x)\}$ for $x,a\in G$, then $a,x\in U(\mu_A;t_2)$. So there exist $y_1\in U(\mu_A;t_2)$ such that $x\in a\cdot y_1$. Also we

have $t_2 \leq \min\{\mu_A(y_1)\}$. Therefore the condition (ii) of Definition 3.1 is verified.
If we put $t_3 = \max\{\lambda_A(a), \lambda_A(x)\}$ then and $a, x \in L(\lambda_A; t_3)$. So there exist $y_2 \in L(\lambda_A; t_3)$ such that $x \in a \cdot y_2$ and we have $\max\{\lambda_A(y_2)\} \leq t_3$, and so the condition (iv) of Definition 3.1 is verified.
This completes the proof.

Corollary 3.9. Let H be an H_v-subgroup of an H_v-group G. If fuzzy sets μ and λ are defined by $\mu(x) = \begin{cases} \alpha_0, & x \in H \\ \alpha_1, & x \in G \setminus H \end{cases}$, $\lambda(x) = \begin{cases} \beta_0, & x \in H \\ \beta_1, & x \in G \setminus H \end{cases}$
Where $0 \leq \alpha_1 < \alpha_0, 0 \leq \beta_0 < \beta$ and $\alpha_i + \beta_i \leq 1$ for $i = 0,1$. Then $A = \{\mu, \lambda\}$ is an intuitionistic fuzzy H_v-subgroup of G and $U(\mu; \alpha_0) = L(\lambda; \beta_0)$.

Corrollary 3.10. Let χ_H be the characteristic function of an H_v-subgroup H of G. Then $A = (\chi_H, \chi_H^c)$ is an intuitionistic fuzzy H_v-subgroup of G.

Theorem 3.11. If $A = \{\mu_A, \lambda_A\}$ is an intuitionistic fuzzy H_v-subgroup of G, then
$\mu_A(x) = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$
And $\lambda_A(x) = \inf\{\alpha \in [0,1] : x \in L(\lambda_A; \alpha)\}$
For all $x \in G$.

Proof Let $\delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0,1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leq \mu_A(x)$ since ε is arbitrary.We now show that $\mu_A(x) \leq \delta$.
If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$ and so
$\beta \in \{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$
Hence
$\mu_A(x) = \beta \leq \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\} = \delta$
Therefore
$\mu_A(x) = \delta = \sup\{\alpha \in [0,1] : x \in U(\mu_A; \alpha)\}$
Now let $\eta = \inf\{\alpha \in [0,1] : x \in L(\lambda_A; \alpha)\}$.
Then $\inf\{\alpha \in [0,1] : x \in L(\lambda_A; \alpha)\} < \eta + \varepsilon$ for any $\varepsilon > 0$ and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0,1]$ with $x \in L(\lambda_A; \alpha)$. Since $\lambda_A(x) \leq \alpha$ and ε is arbitrary, it follows that $\lambda_A(x) \leq \eta$.
To prove $\lambda_A(x) \geq \eta$, let $\lambda_A(x) = \xi$. Then $x \in L(\lambda_A; \xi)$ and thus
$\xi \in \{\alpha \in [0,1] : x \in L(\lambda_A; \alpha)\}$. Hence
$\inf\{\alpha \in [0,1] : x \in L(\lambda_A; \alpha)\} \leq \xi$
i.e. $\eta \leq \xi = \lambda_A(x)$.
Consequently
$\lambda_A(x) = \eta = \inf\{\alpha \in [0,1] : x \in L(\lambda_A; \alpha)\}$
Which completes the proof.

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